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Overlapping sets of priors and the existence of efficient allocations and equilibria for risk measures

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Abstract

The overlapping expectations and the collective absence of arbitrage conditions introduced in the economic literature to insure existence of Pareto optima and equilibria when short-selling is allowed and investors hold a single belief about future returns, is reconsidered. Investors use measures of risk. The overlapping sets of priors and the Pareto equilibrium conditions introduced by Heath and Ku for coherent risk measures are reinterpreted as a weak no-arbitrage and a weak collective absence of arbitrage conditions and shown to imply existence of Pareto optima and Arrow Debreu equilibria.

Keywords: Overlapping sets of priors, collective absence of arbitrage, equilibria with short-selling, risk sharing, measures of risk.

JEL Classification: C62, D50, D81,D84,G1.

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1 Introduction

The problem of the existence and characterization of Pareto optima and equilibria in markets with short-selling, an old problem in the economic literature, has recently been addressed by Barrieu and El Karoui [4], Jouini et al [15], Filipovic and Kupper [9] and Burgert and Rüschendorf [6] for convex measures of risk in infinite markets. Existence of an equilibrium for finite markets where short sales are allowed has first been considered in the early seventies by Grandmont [11], Hart [13], Green [12]. Debreu's standard theorems on existence of equilibrium which assume that the sets of portfolios that investors may hold are bounded below could not be applied. In these early papers, investors were assumed to hold a single homogeneous or heterogeneous probabilistic belief and be von Neumann-Morgenstern (vNM), risk averse utility maximizers. Two sufficient conditions for existence of an equilibrium were given:
- the *overlapping expectations* condition which expresses that investors are sufficiently similar in their beliefs and risk tolerances so that there exists a non empty set of prices (*the no-arbitrage prices*) for which no agent can make costless unbounded vNM utility nondecreasing purchases
-*the no unbounded utility arbitrage* condition, a collective absence of arbitrage condition, which requires that investors do not engage in mutually compatible, utility nondecreasing trades.
These conditions have later been weakened and shown to be equivalent under adequate assumptions and under further assumptions, necessary for existence of equilibrium (see e.g. Page [19], Page and Wooders [20]). They have been generalized to abstract economies (see Werner [24] and Nielsen [17]). Other sufficient conditions were given, in particular that the individually rational utility set (see section 2 for a definition) be compact. For a review of the subject in finite dimension and references, see Allouch et al [1], Dana et al [7], Page [18],[21]. The theory has also been developed for infinite markets. However, since the early work of Kreps [16], it is well known in finance that the concept of arbitrage opportunity used for finite markets is too weak for infinite markets. For finite markets, the no arbitrage condition or the no unbounded utility arbitrage condition insure existence of agents' demand for no-arbitrage prices and the compactness of the set of individually rational attainable allocations (see Nielsen [17]). These conditions break down in infinite dimension (see Brown and Werner [5]). The compactness of the individually rational utility set remains a sufficient condition for existence of equilibria (see Dana et al [8]) but it is in general difficult to provide sufficient conditions on the primitives of an economy to insure that condition.

This paper provides sufficient conditions for existence of Pareto optima and equilibria when agents use convex measures of risk in *finite markets with short-selling*. In contrast, with the papers of Barrieu and El Karoui [4] who deal with families of ρ-dilated risk measures and Jouini et al [15] who consider law invariant convex monetary utilities, it makes no specific assumptions on the risk measures. However it assumes that there is a finite number of states of the world and uses finite dimensional convex analysis techniques. It builds on one hand, on the economic literature on equilibrium with short-selling and on the other hand on a paper by Heath and Ku [14]. Heath and Ku introduced a condition that they called the *Pareto equilibrium*'s condition which requires that if investors do engage in mutually compatible, utility nondecreasing trades, then those trades do not increase their utilities. They showed the equivalence between the *Pareto equilibrium*'s condition and an *overlapping sets of priors* condition (see their proposition 4.2). They however did not address the question of existence of Pareto optima and equilibria. We show that, for risk measures, and for finite markets, the Pareto equilibrium condition coincides with a weakening of the *no unbounded utility arbitrage* condition, that the *overlapping sets of priors* condition is a condition of existence of weak no- arbitrage price and that any of the conditions introduced by Heath and Ku imply existence of Pareto optima and Arrow Debreu equilibria by applying standard results in the theory of equilibrium with consumption sets unbounded below. The same conclusion is also obtained by using a sup-convolution type of technique. Following Heath and Ku, the case of constraints is also studied.

The paper is organized as follows. Section 2 presents the model and recall standard concepts in equilibrium theory. Section 3 defines the concepts of useful and useless trading directions, that of a no-arbitrage price (weak no-arbitrage price) and of collective absence of arbitrage. Two existence of an equilibrium theorems are recalled. Section 4 deals with existence of efficient allocations and equilibria. Sufficient conditions are provided based on one hand on classical economic theory theorems and on the other hand on the sup-convolution. Necessary conditions are also provided. Section 5 deals with the case of constraints on trades.

2 The model

We consider a standard Arrow-Debreu model of complete contingent security markets. There are two dates, 0 and 1. At date 0, there is uncertainty about which state s from a state space $\Omega = \{1, ..., k\}$ will occur at date 1. At date 0, agents trade contingent claims for date 1. The space of contingent claims is the set of random variables from $\Omega \to \mathbb{R}$. The random variable X which equals x_1 in state 1, x_2 in state 2 and x_k in state k, is identified with the vector $X = (x_1, \dots, x_k)$. Let $\triangle = \{\pi \in \mathbb{R}^k_+ : \sum_{s=1}^k \pi_s = 1\}$ be the probability simplex in $\mathbb{R}^k$ and $\pi \in \triangle$. We note $E_\pi(X) := \sum_{l=1}^k \pi_l x_l$ and for $p \in \mathbb{R}^k$, $p \cdot X := \sum_{l=1}^k p_l x_l$.

There are m agents indexed by $i = 1, \dots, m$. Agent i has an endowment $E^i \in \mathbb{R}^k$ of contingent claims. Let $E = \sum_{i=1}^m E^i$ denote aggregate endowment. We assume that each agent has a preference order $\succeq_i$ over $\mathbb{R}^k$ represented by a monetary utility function V^i where we recall that

Definition 1 *A function $V : \mathbb{R}^k \to \mathbb{R}$ is a monetary utility function if it is concave monotone and has the cash invariance property*

$$V(X + C) = V(X) + C, \text{ for any } X \in \mathbb{R}^k, \ C \text{ constant}$$

A positively homogeneous monetary utility function is a monetary utility function that is positively homogeneous.

Monetary utility functions can be identified with *convex measures of risk* (see Föllmer and Schied [10]) and positively homogeneous monetary utility functions with *coherent risk measures*(see Artzner et al [2]) by defining $\rho = -V$. We recall that monetary utility functions have the following representation

$$V(X) = \min_{\pi \in \triangle} E_\pi(X) + c(\pi) \tag{1}$$

where

$$c(\pi) = \sup_{X \in \mathbb{R}^k} V(X) - E_\pi(X) \in \mathbb{R} \cup \{+\infty\} \tag{2}$$

which is convex, lower semi-continuous, is the conjuguate function of V. Let

$$P = \text{dom } c \tag{3}$$

be the set of effective priors associated with V. Clearly, we also have:

$$V(X) = \min_{\pi \in P} E_\pi(X) + c(\pi) \tag{4}$$

Positively homogeneous monetary utility functions are obtained when c is an indicator function δ_P (in other words, $c(\pi) = 0$ if $\pi \in P$ and $c(\pi) = \infty$ otherwise). In that case, $P = \{\pi \in \triangle : c(\pi) = 0\}$ is a convex compact subset of $\triangle$ and we have

$$V(X) = \min_{\pi \in P} E_\pi(X)$$

We next recall standard concepts in equilibrium theory.

An allocation $(X^i)_{i=1}^m \in (\mathbb{R}^k)^m$ is attainable if $\sum_{i=1}^m X^i = E$.
The set of individually rational attainable allocations A is defined by

$$A = \left\{ (X^i)_{i=1}^m \in (\mathbb{R}^k)^m \ \mid \ \sum_{i=1}^m X^i = E \ \text{ and } \ V^i(X^i) \geq V^i(E^i) \text{ for all } \ i \right\}.$$

The individually rational utility set U is defined by

$$U = \left\{ (v_1, v_2, ..., v_m) \in \mathbb{R}^m \mid \exists (X^i)_{i=1}^m \in A \ \text{ s. t. } \ V^i(E^i) \leq v_i \leq V^i(X^i), \ \forall\, i \right\}.$$

Definition 2 *An attainable allocation $(X^i)_{i=1}^m$ is Pareto optimal if there does not exist $(X^{i'})_{i=1}^m$ attainable such that $V^i(X^{i'}) \geq V^i(X^i)$ for all i with a strict inequality for some i. It is individually rational Pareto optimal if it is Pareto optimal and $V^i(X^i) \geq V^i(E^i)$ for all i.*

Definition 3 *A pair $(X^*, p^*) \in A \times \mathbb{R}^k \backslash \{0\}$ is a contingent Arrow-Debreu equilibrium if*

1. *for each agent i and $X^i \in \mathbb{R}^k$, $V^i(X^i) > V(X^{*i})$ implies $p^* \cdot X^i > p^* \cdot X^{*i}$,*
2. *for each agent i, $p^* \cdot X^{*i} = p^* \cdot E^i$.*

Assertions 1 and 2 express that X^{*i} solves investor's i maximization problem at price p^*. Markets clear since X^* is attainable.

We shall use the following notations. Given a subset $A \subseteq \mathbb{R}^p$, int A is the interior of A, cl A is the closure of A. For A convex, ri A is the relative interior of A, $A^0 = \{p \in \mathbb{R}^p \ \mid \ p \cdot X \leq 0, \ \text{ for all } \ X \in A\}$ is the polar of A.

3 Useful vectors and no-arbitrage concepts

In this section, we recall a number of standard concept in the theory of equilibrium with short-selling. We first define and characterize for monetaries utilities the useful (useless) trading direction. We next define the concept of a no-arbitrage price (weak no-arbitrage price) as a price giving strictly positive value to any useful non zero vector (useful and not useless vector). We then recall concepts of collective absence of arbitrage and two existence of an equilibrium theorems.

3.1 Useful vectors

Let V be a monetary utility of type (4). For $X \in \mathbb{R}^k$, let

$$\widehat{P}(X) = \{Y \in \mathbb{R}^k \mid V(Y) \geq V(X)\}$$

be the set of contingent claims preferred to X and $R(X)$ be its asymptotic cone (see Rockafellar [22], section 8). Since V is concave, by Rockafellar's theorem 8.7 in [22], $R(X)$ is independent of X. It will be simply denoted by R. It is called the set of *useful vectors* for V in the economic literature. We recall that

$$R = \{W \in \mathbb{R}^k \mid V(\lambda W) \geq V(0), \text{ for all } \lambda \geq 0\}.$$

The lineality space of V or set of useless vectors is defined by

$$L = \{W \in \mathbb{R}^k \, | V(\lambda W) \geq V(0), \text{ for all } \lambda \in \mathbb{R}\} = R \cap (-R).$$

We first characterize R.

Proposition 1 *We have*

$$R = \{W \in \mathbb{R}^k \ \mid E_\pi(W) \geq 0, \ \textit{for all}\, \pi \in P\}$$

$$L = \{W \in \mathbb{R}^k \ \mid E_\pi(W) = 0, \ \textit{for all}\, \pi \in P\}$$

There exists $\pi \in$ int P if and only if $L = \{0\}$.

Proof: Let W fulfill $E_\pi(W) \geq 0$ for all $\pi \in P$. Then

$$V(\lambda W) = \min_{\pi \in P} E_\pi(\lambda W) + c(\pi) \geq \min_{\pi \in P} c(\pi) = V(0) \text{ for all } \lambda \geq 0$$

which implies that $W \in R$. Conversely, let $W \in R$. Then

$$V(\lambda W) \geq V(0), \text{ for all } \lambda \geq 0,$$

hence $E_\pi(\lambda W) + c(\pi) \geq V(0)$, for all $\lambda \geq 0$, $\pi \in P$. For a fixed $\pi \in P$, the map $\lambda \to \lambda E_\pi(W)$ is bounded below, hence $E_\pi(W) \geq 0$. The proofs of the other assertions are straightforward. ■

For further use, let R^i be the set of useful vectors corresponding to V^i.

3.2 Concepts of absence of arbitrage

The first concept that we recall is that of a no-arbitrage price, a price for which no agent can make costless unbounded utility nondecreasing purchases.

Definition 4 *A price vector $p \in \mathbb{R}^k$ is a " no-arbitrage price" for i if $p \cdot W > 0$, for all $W \in R^i \backslash \{0\}$. A price vector $p \in \mathbb{R}^k$ is a " no-arbitrage price" for the economy if it is a no-arbitrage price for each agent.*

Let S^i denote the set of no arbitrage prices for i. Then $S^i = \text{int} - (R^i)^0$. A price vector $p \in \mathbb{R}^k$ is a " no-arbitrage price" for the economy if and only if $p \in \cap_i S^i = \cap_i \text{int} - (R^i)^0$.

From Rockafellar's [22] corollary 14.6.1, $S^i \neq \emptyset$ if and only if $L^i = \{0\}$. Therefore, there exists a no-arbitrage price for the economy only if $L^i = \{0\}$ for all i. This leads us to introduce a weaker no-arbitrage concept due to Werner [24].

Definition 5 *A price vector $p \in \mathbb{R}^k$ is a " weak no-arbitrage price" for agent i if $p \cdot W > 0$ for all $W \in R^i \backslash L^i$. A price vector $p \in \mathbb{R}^k$ is a "weak no-arbitrage price" for the economy if it is a weak no-arbitrage price for each agent.*

If p is a weak no-arbitrage price for i, then for every $W \in R^i \cap (L^i)^\perp \backslash \{0\}$ and $W' \in L^i$, $\alpha W + \beta W' \in R^i \backslash L^i$ for every $\alpha > 0,\ \beta \in \mathbb{R}$. Hence $p \cdot (\alpha W + \beta W') = \alpha p \cdot W + \beta p \cdot W' > 0$ for every $\alpha > 0,\ \beta \in \mathbb{R}$. Therefore $p \cdot W' = 0$ for any $W' \in L^i$. In other words a "weak no-arbitrage price" for i gives 0 value to any useless trade for i. Hence p is a " weak no-arbitrage price" for agent i if $p \in (L^i)^\perp$ and $p \cdot W > 0$ for all $W \in (R^i \cap (L^i)^\perp) \backslash \{0\}$. The converse is trivially true. Let S^i_w denote the set of weak no arbitrage prices for i. We have the following characterization of S^i_w.

Lemma 1 *The set of weak no-arbitrage prices for i, $S^i_w = ri - (R^i)^0$. Hence the set of weak no arbitrage prices for the economy is $\cap_i S^i_w = \cap_i ri(-(R^i)^0)$.*

For a proof, the reader is refered to Allouch et al [1], lemma 2.

Let us now characterize the set of no-arbitrages prices and weak no-arbitrage prices.

Proposition 2 *Let V^i fulfill (4) for each i. Then*

1. *the set of non arbitrage prices for agent i is $S^i =$ cone int P^i,*
2. *the set of non arbitrage prices for the economy is $\underset{i}{\cap} S^i =$ cone $\underset{i}{\cap}$ int P^i,*
3. *the set of weak no arbitrage prices for agent i is $S^i_w =$ cone ri P^i*
4. *the set of weak no arbitrage prices for the economy is $\cap_i S^i_w =$ cone $\cap_i$ ri P^i*

Proof: From proposition 1, $R^i = \{W \in \mathbb{R}^k \mid E_\pi(W) \geq 0,\ \text{for all } \pi \in P^i\}$, hence the set of no-arbitrage prices for i, $S^i = \text{int cl cone } P^i$. Since cone P^i is convex, int cl cone $P^i =$ int cone P^i and

$$S^i = \text{ int cone } P^i = \text{ cone int } P^i. \tag{5}$$

The set of no-arbitrage prices for the economy

$$\bigcap_i S^i = \bigcap_i \text{int cone } P^i = \text{ cone } \bigcap_i \text{int } P^i \tag{6}$$

From lemma 1 and from Rockafellar's [22] theorem 6.3 and corollary 6.6.1, the set of weak no arbitrage prices for i,

$$S_w^i = \text{ri } -(R^i)^0 = \text{ri cl cone } P^i = \text{ri cone } P^i = \text{cone ri} P^i \tag{7}$$

Hence the set of weak no arbitrage prices for the economy is

$$\cap_i S_w^i = \cap_i \text{ri } -(R^i)^0 = \cap_i \text{ri(cone} P^i) = \cap_i \text{cone (ri} P^i) = \text{cone } \cap_i \text{ ri} P^i \tag{8}$$

■

We now turn to concepts of *collective absence of arbitrage.*

Let us first recall the *no-unbounded-arbitrage* condition denoted now on by NUBA introduced by Page [19] which requires inexistence of an unbounded set of mutually compatible net trades which are utility nondecreasing.

Definition 6 *The economy satisfies the NUBA condition if* $\sum_i W^i = 0$ *and* $W^i \in R^i$ *for all* i*, implies* $W^i = 0$ *for all* i*.*

Let us also recall a weaker condition, called the Weak No-Market-Arbitrage condition, introduced by Hart [13] which requires that all mutually compatible net trades which are utility nondecreasing be useless.

Definition 7 *The economy satisfies the Weak No-Market-Arbitrage condition (WNMA) if* $\sum_i W^i = 0$ *and* $W^i \in R^i$ *for all* i *implies* $W^i \in L^i$*, for all* i*.*

The following proposition follows directly from proposition 1.

Proposition 3 *Let* V^i *fulfill (4) for each* i*. Then*

1. *the economy satisfies NUBA if there exists no set of net trades* $W^1, \ldots, W^n$ $W^i \neq 0$ *for all* i *with* $E_\pi(W^i) \geq 0$ *for all* $\pi \in P^i$ *and all* i *and* $\sum_i W^i = 0$*,*

2. *the economy satisfies WNMA if there exists no set of net trades* $W^1, \ldots, W^n$ *with* $E_\pi(W^i) \geq 0$ *for all* $\pi \in P^i$ *and all* i *with a strict inequality for some* i *and* $\pi \in P^i$ *and* $\sum_i W^i = 0$*.*

WNMA has been introduced by Heath and Ku [14] under the name of *Pareto equilibrium.*

3.3 Existence of equilibrium theorems

We end this section by recalling two theorems obtained in the literature on equilibrium with short-selling:

Theorem 1 *Let V^i fulfill (4) for each i. Then the following assertions are equivalent*

1. $\bigcap_i S^i \neq \emptyset$
2. *NUBA is fulfilled,*
3. *the set of individually rational attainable allocations A is compact.* *Any of the previous assertion imply any of the following assertions:*
4. *The individually rational utility set U is compact,*
5. *there exists an individually rational Pareto optimal allocation,*
6. *there exists an equilibrium.*

Proof: See e.g. Page and Wooders [20], Dana et al [7]. ■

Theorem 2 *Let V^i fulfill (4) for each i. Then the following equivalent assertions are equivalent.*

1. $\bigcap_i S^i_w \neq \emptyset$
2. *WNMA is fulfilled.*

Any of the previous assertions imply any of the following assertions:

1. *the individually rational utility set U is compact,*
2. *there exists a Pareto optimal allocation,*
3. *there exists an equilibrium.*

Proof: See e.g. Page et al [21], Allouch et al [1]. ■

Theorem 1 make stronger requirements than theorem 2, which are equivalent to assuming that the set of individually rational attainable allocations is compact. It follows from theorem 2 by assuming $L^i = \{0\}$ for all i. On can show (see Allouch et al [1]) that the requirements of theorem 2 are equivalent to assuming that the projection of the set of individually rational attainable allocations onto $\Pi_{i=1}^m (L^i)^\perp$ is compact. Finally when the utilities are strictly concave, then all the conditions of theorem 1 are equivalent.

4 Existence of efficient allocations and Equilibria

4.1 Sufficient conditions for existence of efficient allocations and equilibria

Next proposition follows from theorem 1 and propositions 2 and 3.

Proposition 4 *Let V^i fulfill (4) for each i. Then the following are equivalent:*

1. $\bigcap\limits_i int P^i \neq \emptyset$

2. *there exists no set of net trades $W^1, \ldots, W^n$, $W^i \neq 0$ for all i with $E_\pi(W^i) \geq 0$ for all $\pi \in P^i$ and all i and $\sum_i W^i = 0$,*

3. *the set of individually rational attainable allocations A is compact.*
   Any of the previous assertions imply any of the following assertions:

4. *there exists an individually rational Pareto optimal allocation,*

5. *there exists an equilibrium.*

In order to give an economic interpretation of Heath and Ku's Pareto equilibrium condition, let $P^i = \text{dom } c^i$ be the set of effective priors associated with V^i a monetary utility for agent i. Consider the following incomplete preferences on pairs $(X, Y) \in \mathbb{R}^k \times \mathbb{R}^k$ defined by

$$X \succeq_{P^i} Y \quad \text{iff} \quad E_\pi(X) \succeq E_\pi(Y) \text{ for all } \pi \in P^i \tag{9}$$

Given $\mathcal{P} = (P^i)_{i=1}^m$ the set of effective priors of the m agents, an attainable allocation $(X^i)_{i=1}^m$ is P-*Pareto optimal* if there does not exist a set of net trades $(W^i)_{i=1}^m$ with $\sum_i W^i = 0$ such that $E_\pi(X^i + W^i) \geq E_\pi(X_i)$ for all i and all $\pi \in P^i$ with a strict inequality for some i and some $\pi \in P^i$. Equivalently there does not exist a set of net trades $(W^i)_{i=1}^m$ with $\sum_i W^i = 0$ such that $E_\pi(W^i) \geq 0$ for all i and all $\pi \in P^i$ with a strict inequality for some i and some $\pi \in P^i$. Hence, for the incomplete preference defined by (9), either any attainable allocation is P-Pareto optimal or no attainable allocation is P-Pareto optimal.
The next theorem which follows from theorem 2 and propositions 2 and 3 may be viewed as an alaboration of Heath and Ku's [14] proposition 4.2 which shows the equivalence between existence of a weak no-arbitrage price and WNMA. It establishes sufficient conditions for existence of a Pareto allocation or equivalently of an equilibrium for monetary utilities.

Theorem 3 *Let V^i fulfill (4) for each i. Then the following are equivalent:*

1. $\bigcap\limits_i ri\, P^i \neq \emptyset$,

2. *there exists no set of net trades* $W^1, \ldots, W^n$*, with* $E_\pi(W^i) \geq 0$ *for all* $\pi \in P^i$ *and all* i *with a strict inequality for some* i *and* $\pi \in P^i$ *and* $\sum_i W^i = 0$*,*

3. *any attainable allocation is P-Pareto optimal.*
   Any of the previous assertions imply any of the following assertions:

4. *there exists an individually rational Pareto optimal allocation,*

5. *there exists an equilibrium.*

Corollary 1 *Let* V^i *fulfill (4) for each* i*. If* P^i *is independent of* i*, then there exists an individually rational Pareto optimal allocation and there exists an equilibrium.*

Proof: Let P be the common set of probabilities and S_w be the set of weak no-arbitrage price. Since P is convex, ri $P \neq \emptyset$. From proposition 6 assertion 2, $S_w \neq \emptyset$. ■

4.2 Necessary conditions for existence of equilibria

Proposition 6 and theorem 3 provide sufficient conditions for existence of efficient allocations (or of an equilibrium). We next give necessary conditions for existence of an efficient allocations.

Proposition 5 *Let* V^i *fulfill (4) for each* i*. If there exists an efficient allocations, then*

1. $\bigcap\limits_i P^i \neq \emptyset$*,*

2. *there does not exist a set of trades* $W^1, \ldots, W^n$ *fulfilling* $E_\pi(W^i) > 0$ *for all* $\pi \in P^i$ *and for all* i *and* $\sum_i W^i = 0$*,*

3. *the individually rational utility set* U *is bounded,*

4. *If* $(p^*, (X^{*i})_{i=1}^m)$ *is an equilibrium, then* $p^* \in \bigcap\limits_i P^i$ *and* $p^* \cdot W > 0$ *for any* W *fulfilling* $E_\pi(W) > 0$ *for all* $\pi \in P^i$ *for some* i*.*

Proof: We first prove assertion one. Since there exists an equilibrium (X^*, p^*), for every i, there exists $\lambda^i > 0$, such that $p^* \in \lambda^i \delta V^i(X^{*i})$. As $\delta V^i(X^{*i}) \subseteq P^i$, for each i, there exists $\pi^i \in P^i$ such that $\lambda^i \pi^i$ is independent of i. Hence λ^i and π^i are independent of i and $\pi \in \bigcap\limits_i P^i \neq \emptyset$ as was to be proven. The second assertion is obvious. To prove the third assertion, for any $(z^i) \in U$, there exists $(X^1, X^2, ..., X^m) \in A$ such that

$$V^i(E^i) \leq z^i \leq V^i(X^i), \text{for all } i. \tag{10}$$

Since $\underset{i}{\cap} P^i \neq \emptyset$, let $\pi \in \underset{i}{\cap} P^i$. We have $V^i(X^i) \leq E_\pi(X^i) + c^i(\pi)$, thus, from (10),

$$m^i = V^i(E^i) - c^i(\pi) \leq E_\pi(X^i).$$

Since for all i, $E_\pi(X^i)$ is bounded below by m^i, it is bounded above by $M^i = E_\pi(E) - \sum_{l \neq i} m^l$. From (10), we thus have

$$V^i(E^i) \leq z^i \leq V^i(X^i) \leq M^i + c^i(\pi) \text{ for all } i$$

and U is bounded. To prove the last assertion, clearly $p^* \in \underset{i}{\cap} P^i$. Let W fulfill $E_\pi(W) > 0$ for all $\pi \in P^i$ for some i. We then have

$$V^i(X^{*i} + W) > V^i(X^{*i}),$$

hence by definition of an equilibrium $p^* \cdot W > 0$. ■

Remark 1

1. *Assertion one is weaker than the sufficient condition $\underset{i}{\cap}$ ri $P^i \neq \emptyset$. Assertion 4 is weaker than the weak no-arbitrage price condition. We see that there is a gap between the necessary and the sufficient conditions.*
2. *If V^i is coherent for each i, then P^i is convex compact, for each i. In that case, it follows from Samet [23] that $\underset{i}{\cap} P^i \neq \emptyset$ is equivalent to the assertion: there does not exist a set of trades $W^1, \ldots, W^n$ fulfilling $E_\pi(W^i) > 0$ for all $\pi \in P^i$ and for all i and $\sum_i W^i = 0$.*
3. *Assertions 1 and 3 generalize to the infinite dimension.*

Let us apply the previous proposition to the case where P^i reduces to a unique probability for all i .

Corollary 2 *Let V^i fulfill (4) for each i. Let $P^i = \{\pi^i\}$ for all i. Then there exists an equilibrium if and only if π^i is independent of i.*

Proof: The sufficient condition follows from corollary 1 while the necessary condition from proposition 5. ■

4.3 Sup-convolution

We end this section by providing an alternative approach based on the sup-convolution used by Barrieu and El Karoui [4], Filipovic and Kupper [9], Jouini et al [15], Burgert and Rüschendorf [6] in an infinite dimensional framework.

As is well known, from the monetary invariance, an attainable allocation is Pareto optimal for aggregate endowment E if and only if it solves the following problem:

$$\begin{cases} \sup \sum_{i=1}^{m} V^i(X^i) \text{ subject to} \\ \sum_{i=1}^{m} X^i = E. \end{cases}$$

For $X \in \mathbb{R}^k$, let

$$\Box_i V^i(X) = \sup \left\{ \sum_{i=1}^{m} V^i(X^i), \ \sum_{i=1}^{m} X^i = X \right\}$$

be the sup-convolution of the V^i. Since V^i is finite for every i, $\Box_i V^i(X) > -\infty$ and dom $\Box_i V^i = \mathbb{R}^k$ if and only if $\bigcap_i \text{dom } c^i = \bigcap_i P^i \neq \emptyset$. In that case, $\Box_i V^i$ is a monetary utility (the representative agent utility when aggregate endowment is X) and $\Box_i V^i$ and $\sum_{i=1}^{m} c^i$ are conjuguate. Furthermore, from Rockafellar's theorem 16.4 [22], a sufficient condition for existence of a Pareto optimum $(X^1, \ldots, X^m)$ is that

$$\bigcap_i \text{ri dom } c^i = \bigcap_i \text{ ri } P^i \neq \emptyset. \tag{11}$$

We thus reobtain by the sup-convolution approach, the Heath and Ku's proposition 4.2 sufficient condition for existence of a Pareto optimum.

Furthermore, let us show directly that (11) is a sufficient condition for existence of an equilibrium.
Let us first remark that $\pi \in \partial \Box_i V^i(X)$ iff $\pi \in \bigcap_i \partial V^i(X^i)$ for any Pareto optimum $(X^1, \ldots, X^m)$ associated with X. Indeed,

$$\pi \in \partial \Box_i V^i(X) \text{ iff } \Box_i V^i(X) = \sum_{i=1}^{m} c^i(\pi) + E_\pi(X).$$

Since $\Box_i V^i(X) = \sum_{i=1}^{m} V^i(X^i)$ for any Pareto optimum $(X^1, \ldots, X^m)$ associated with X and $c^i(\pi) + E_\pi(X^i) - V^i(X^i) \geq 0$, for all $\pi \in \triangle$, we obtain that

$$V^i(X^i) = c^i(\pi) + E_\pi(X^i) \ \text{ for all } \ i, \ \text{ equivalently, } \pi \in \bigcap_i \partial V^i(X^i)$$

Therefore, a pair $((X^{*i})_{i=1}^m, p^*) \in A \times \mathbb{R}^k \setminus \{0\}$ is a contingent Arrow-Debreu equilibrium, when aggregate endowment is E iff

1. $(X^{*i})_{i=1}^m$ is Pareto optimal,

2. $p^* \in \lambda \partial \Box_i V^i(E)$ for some $\lambda > 0$,

3. $p^* \cdot X^{*i} = p^* \cdot E^i$ for all i.

As remarked by Filipovic and Kupper [9], given a Pareto optimum $(X^1, \ldots, X^m)$, we may construct equilibria without using a fixed point theorem, because of the axiom of monetary invariance. Indeed, given any $p \in \lambda\partial\Box_i V^i(E)$, we claim that $(X^1 + p\cdot(E^1 - X^1), \ldots, X^m + p\cdot(E^m - X^m), p)$ is an equilibrium. Indeed, $(X^1 + p\cdot(E^1 - X^1), \ldots, X^m + p\cdot(E^m - X^m))$ is Pareto optimal as

$$\sum_{i=1}^m V^i(X^i + p\cdot(E^i - X^i)) = \sum_{i=1}^m V^i(X^i) + \sum_{i=1}^m p\cdot(E^i - X^i) = \sum_{i=1}^m V^i(X^i)$$

since $(X^1, \ldots, X^m)$ is attainable. By construction p fulfills assertion 2 and $p\cdot(X^i + p\cdot(E^i - X^i)) = p\cdot E^i$ for all i.

5 Constraints on exchanges

5.1 The model

Constraints on exchanges when agents use measures of risk, have already be discussed by Heath and Ku [14], Filipovic and Kupper [9] and Burgert and Rüschendorf [6].

We now assume that trades are only possible in linear subspaces $M^i \subseteq \mathbb{R}^k$, $1 \le i \le k$. Agent i has an endowment $E^i \in M^i$ of contingent claims. The definitions of attainable, individual rational and Pareto optimal allocations and equilibria are extended by imposing the constraint that $X^i \in M^i$ for all i. In particular, the set of constrained useful vectors for i is defined as

$$R^{M^i} = \{W \in M^i \mid V(\lambda W) \ge V(0),\ \text{for all } \lambda \ge 0\}.$$

Therefore

$$R^{M^i} = \{W \in M^i \mid E_\pi(W) \ge 0,\ \text{for all } \pi \in P\} = R^i \cap M^i$$

where R^i is the unconstrained set of useful vectors for i.

5.2 Weak no-arbitrage prices under constraints

In order to characterize weak no-arbitrage prices for this new economy, let us first characterize $R^0_{M^i}$ the polar of the set of constrained useful vectors for i. From Rockafellar's corollary 16.4.2.,

$$(R^{M^i})^0 = \mathrm{cl}((R^i)^0 + (M^i)^\perp)$$

and from Rockafellar's theorem 6.3 and corollary 6.6.2.

$$\text{ri } (R^{M^i})^0 = \text{ ri cl}((R^i)^0 + (M^i)^\perp) = \text{ri } ((R^i)^0 + (M^i)^\perp) = \text{ri } (R^i)^0 + (M^i)^\perp$$

Using Rockafellar's corollary 6.6.2., we obtain that

$$\cap_i S_w^i = \cap_i \text{ri } (R^{M^i})^0 = \cap_i(\text{ricone}P^i + (M^i)^\perp) = \cap_i \text{cone } (\text{ri}P^i + (M^i)^\perp)$$

hence,

$$\cap_i S_w^i \neq \emptyset \text{ iff } \cap_i \text{cone } (\text{ri}P^i + (M^i)^\perp) \neq \emptyset \tag{12}$$

Since (12) is positively homogeneous, let $H = \{m \in \mathbb{R}^k \mid \sum_j m_j = 1\}$. The set of weak no-arbitrage price is non empty if and on if there exists $\mu \in H$ such that, for any i

$$\mu = \lambda^i \pi^i + m_\perp^i$$

with $\pi^i \in$riP^i and $\lambda^i > 0$ and $m_\perp^i \in (M^i)^\perp$. The vector μ may be interpreted as a signed measure and we have

$$E_\mu(X^i) = \lambda^i E_{\pi^i}(X^i), \text{ for all } \ X^i \in M^i \text{ and } \ i \tag{13}$$

with $\pi^i \in$riP^i, $\lambda^i > 0$. Hence the restriction of μ to M^i is a non-negative measure proportionnal to a prior in the relative interior of P^i. Furthermore,

- if agent i can trade the riskless asset or equivalently if constants belong to M^i, then $\lambda^i =< \mu, 1 >= 1$.
- If all agents can trade the riskless asset, then λ^i is independent of i. (13) may be rewritten as: there exists a signed measure μ and probabilities π^i in the relative interior of P^i for each agent such that

$$E_\mu(X^i) = E_{\pi^i}(X^i), \text{ for all } \ X^i \in M^i \text{ and } \ i \tag{14}$$

- If all agents can trade the riskless asset and if $M^i = \mathbb{R}^k$ for some i, then μ is a probability measure and (13) holds true.

Remark 2 *1. Condition (12) is equivalent to the WNMA condition: there exists no set of net trades $W^1, \ldots, W^n$, with $W^i \in M^i$ for all i and $E_\pi(W^i) \geq 0$ for all $\pi \in P^i$ and all i with a strict inequality for some i and $\pi \in P^i$ and $\sum_i W^i = 0$.*

2. *The condition $\mu = \lambda^i \pi^i + m_\perp^i$ for all i is very similar to the condition one obtains when writing the no-arbitrage condition for finite financial markets with constraints on portfolios.*

3. *We next show that we cannot dispense with the constants λ_i in (13).*

   Example 1

   There are two states and three agents. Each of them has a unique probability over states: agent 1 has probability $\pi^1 = (\frac{1}{4}, \frac{3}{4})$, agent 2 probability

$\pi^2 = (\frac{3}{4}, \frac{1}{4})$ and agent 3, probability $\pi^3 = (1, 0)$. Assume that the trading sets are:

$$\begin{aligned} M^1 &= \{X^1 = (x^1, x^1) \mid x^1 \in \mathbb{R}\} \\ M^2 &= \{X^2 = (x^2, -x^2) \mid x^2 \in \mathbb{R}\} \\ M^3 &= \{X^3 = (x^3, 0) \mid x^3 \in \mathbb{R}\}. \end{aligned}$$

The WNMA condition is fulfilled (or equivalently, there exists a Heath and Ku Pareto equilibrium) since $R^{M^1} = \{W^1 = (w, w) \mid w \geq 0\}$, $R^{M^2} = \{W^2 = (w, -w) \mid w \geq 0\}$ and $R^{M^3} = \{W^3 = (w, 0) \mid w \geq 0\}$ and $\sum_i W^i = 0$ implies $W^i = 0$ for all i. But there exists no solution $\mu = (\mu_1, \mu_2)$ to the following system:

$$\begin{aligned} E_\mu(X^1) = (\mu_1 + \mu_2)x^1 &= E_{\pi^1}(X^1) = x^1, \text{ for all } x^1 \in \mathbb{R}, \\ E_\mu(X^2) = (\mu_1 - \mu_2)x^2 &= E_{\pi^2}(X^2) = 1/2x^2, \text{ for all } x^2 \in \mathbb{R}, \\ E_\mu(X^3) = \mu_1 x^3 &= E_{\pi^3}(X^3) = x^3, \text{ for all } x^3 \in \mathbb{R}. \end{aligned}$$

since the first and the third equations imply $\mu_1 = 1$, $\mu_2 = 0$ which is incompatible with the second equation.

Example 2

There are three agents. The state space and the probabilities are as in Example 1 as well as the trading sets of agents 1 and 2. The trading set of agent 3 is $M^3 = \mathbb{R}^2$. Hence, $R^{M^3} = \{W^3 = (w_1^3, w_2^3) \mid w_1^3 \geq 0\}$. As in the previous example, the WNMA condition is fulfilled. However, there exists no solution $\mu = (\mu_1, \mu_2)$ to the following system:

$$\begin{aligned} E_\mu(X^1) &= (\mu_1 + \mu_2)x^1 = & E_{\pi^1}(X^1) = x^1, \text{ for all } X^1 \in M^1, \\ E_\mu(X^2) &= (\mu_1 - \mu_2)x^2 = & E_{\pi^2}(X^2) = 1/2x^2, \text{ for all } X^2 \in M^1, \\ E_\mu(X^3) &= \mu_1 x_1^3 + \mu_2 x_2^3 = & E_{\pi^3}(X^3) = x_1^3, \text{ for all } X^3 \in M^3. \end{aligned}$$

since the third equation implies $\mu_1 = 1$, $\mu_2 = 0$ which is incompatible with the second equation.

4. *Finally, we give an example where the agents have the same trading set which contains the riskless asset but the intersection of the sets of priors is empty.*

   Example 3

   The state space and the probabilities are as in Example 1, the trading sets are

$$M^1 = M^2 = M^3 = \{(x, x) \mid x \in \mathbb{R}\}.$$

   Here $R^{M^1} = R^{M^2} = R^{M^3} = \{(w, w) \mid w \geq 0\}$. Hence, the WNMA condition is fulfilled but $\cap_i P^i = \emptyset$.

Let us summarize the results obtained in a proposition:

Proposition 6 *Let V^i fulfill (4) and agent's i trading set be the subspace M^i for each i. Then the following are equivalent:*

1. *there exists a signed measure μ and positive constants λ^i and probabilities $\pi^i \in riP^i$ such that (13) holds true,*
2. *there exists no set of net trades $W^1, \ldots, W^n$, with $W^i \in M^i$ for all i and $E_\pi(W^i) \geq 0$ for all $\pi \in P^i$ and all i with a strict inequality for some i and $\pi \in P^i$ and $\sum_i W^i = 0$.*
   Any of the previous assertions implies the existence of efficient allocations or of an equilibrium.

5.3 Sup-convolution and constraints

Assuming to simplify that agents can all trade the riskless asset, let us return to the inf-convolution's approach. Define for each i

$$V^{M^i}(X) = \begin{cases} V^i(X) \text{ if } X \in M^i \\ -\infty, \text{ otherwise} \end{cases} \tag{15}$$

The function $V^{M^i} : \mathbb{R}^k \to \mathbb{R} \cup \{-\infty\}$ is concave, upper semi-continuous, cash invariant but fails to be monotone. We may still use duality methods but the domain of the conjuguate function is larger than the probability simplex. Let $m \in \mathbb{R}^k$ and

$$c^{M^i}(m) = \sup_{X \in \mathbb{R}^k} V^{M^i}(X) - < m, X >= \sup_{X \in M^i} V^{M^i}(X) - < m, X > \tag{16}$$

be the conjuguate of V^{M^i}. Clearly we have

$$c^{M^i}(m + m^\perp) = c^{M^i}(m), \text{ for all } m^\perp \in (M^i)^\perp \tag{17}$$

From the cash invariance of V^{M^i}, we also have

$$c^{M^i}(m) = \sup_{X \in M^i,\, a \in \mathbb{R}} V^{M^i}(X) - < m, X > +a(1 - < m, 1 >)$$

therefore $c^{M^i}(m) = \infty$ if $1 \neq < m, 1 >$. Defining $H = \{m \in \mathbb{R}^k \mid \sum_j m_j = 1\}$, we thus have that dom $c^{M^i} \subseteq H$.
For $m \in P^i$, $c^{M^i}(m) \leq c^i(m) < \infty$. Hence

$$\text{dom } c^{M^i} = (P^i + (M^i)^\perp) \cap H$$

The function $\Box_i V^{M^i} < \infty$ if and only if $\bigcap_i \text{dom } c^{M^i} = \bigcap_i (P^i + (M^i)^\perp) \cap H \neq \emptyset$. In that case, since $\Box_i V^{M^i} > -\infty$ on $\sum_i M^i$, dom $\Box_i V^{M^i} \neq \emptyset$ and $\Box_i V^{M^i}$ is proper, hence $\Box_i V^{M^i}$ and $\sum_{i=1}^m c^{M^i}$ are conjuguate. From Rockafellar's theorem 16.4 [22], a sufficient condition for existence of a Pareto optimum $(X^1, \ldots, X^m)$ is that

$$\bigcap_i \text{ri dom } c^{M^i} = \bigcap_i (\text{ri } (P^i) + (M^i)^{\perp}) \cap H \neq \emptyset. \tag{18}$$

We are thus back to the weak no-arbitrage condition (12).

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